Chapter 5

Review of Series Solutions, System DE. and Stability

5.1 Solutions about Ordinary Points

5.1.1 Power Series Solution

Series solution method is useful when the coefficients are not constant or when methods introduced in the previous sections does not work. For example, we have a **Bessel equation**

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \quad (\nu \geq 0)$$

or **Legendre equation**

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0, \quad (\alpha \geq 0).$$

Example 5.1.1. Find expansion of $\frac{1}{x^2-2x+5}$ at $x = 1$. What is radius of convergence ?

Sol. For $\left|\frac{x-1}{2}\right| < 1$

$$\begin{aligned}\frac{1}{x^2 - 2x + 5} &= \frac{1}{(x-1)^2 + 4} = \frac{1}{4}\frac{1}{1 + \left(\frac{x-1}{2}\right)^2} \\ &= \frac{1}{4}\left(1 - \left(\frac{x-1}{2}\right)^2 + \left(\frac{x-1}{2}\right)^4 + \cdots + (-1)^n\left(\frac{x-1}{2}\right)^{2n} + \cdots\right).\end{aligned}$$

Hence radius of convergence is 2.

□

Example 5.1.2. Solve $(x^2+1)y''+xy'-y=0$.

Solution. This equation has a singularity at $x=\pm i$ and power series will converge for $|x|<1$ only. With $y(x)=\sum_{n=0}^{\infty} c_n x^n$ we find

$$
\begin{aligned}
&(x^2+1)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x\sum_{n=1}^{\infty} nc_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n \\
&= \sum_{n=2}^{\infty} n(n-1)c_n x^n + \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} nc_n x^n - \sum_{n=0}^{\infty} c_n x^n \\
&= -c_0 + \sum_{k=2}^{\infty} k(k-1)c_k x^k + \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k + \sum_{k=1}^{\infty}[kc_k - c_k]x^k \\
&= 2c_2 - c_0 + 6c_3 x + \sum_{k=2}^{\infty}[k(k-1)c_k + (k+2)(k+1)c_{k+2} + kc_k - c_k]x^k \\
&= 2c_2 - c_0 + 6c_3 x + \sum_{k=2}^{\infty}[(k+1)(k-1)c_k + (k+2)(k+1)c_{k+2}]x^k = 0.
\end{aligned}
$$

Comparing the coefficients, we see $2c_2 - c_0 = 0,\ c_3 = 0$ and

$$(k+1)(k-1)c_k + (k+2)(k+1)c_{k+2}, \quad k = 2, 3, \cdots. \tag{5.1}$$

Thus,

$$
\begin{aligned}
c_2 &= \frac{1}{2}c_0, \quad c_3 = 0 \\
c_{k+2} &= \frac{1-k}{k+2}c_k, k = 2, 3, \cdots
\end{aligned}
$$

Hence

$$
\begin{aligned}
c_4 &= -\frac{1}{4}c_2 = -\frac{1}{2\cdot 4}c_0 = -\frac{1}{2^2\cdot 2!}c_0 \\
c_5 &= -\frac{2}{5}c_3 = 0 \\
c_6 &= -\frac{3}{6}c_4 = \frac{3}{2\cdot 2\cdot 4\cdot 6}c_0 = \frac{1\cdot 3}{2^3\cdot 3!}c_0 \\
c_7 &= -\frac{4}{7}c_5 = 0 \\
&= \cdots.
\end{aligned}
$$

Note that there is no conditions or relation on c_1(free). So $y = c_1 x$ is a solution. Grouping terms of c_0 and c_1 we have the solution $y = c_0 y_1 + c_1 y_2$:

$$y_1 = 1 + \frac{1}{2}x^2 + \sum_{n=2}^{\infty}(-1)^{n-1}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n n!}x^{2n},\ y_2 = x,\ \text{for } |x| < 1.$$

5.2 Solution near Singular Points

Definition 5.2.1.

$$= \quad (x-x_0)^2 y'' + (x-x_0)p(x)y' + q(x)y = 0, \tag{5.2}$$

where $p(x)$ and $q(x)$ are analytic.

Frobenius Method

Theorem 5.2.2 (Frobenius(1849-1917) Theorem)**.** *If x_0 is a regular singular point, then there exists at least one nonzero solution of the form*

$$y(x) = (x-x_0)^r \sum_{n=0}^{\infty} c_n (x-x_0)^n, \quad a_0 \neq 0, \tag{5.3}$$

where r(not nec. an integer) is a constant to be determined.

Example 5.2.3. [Distinct roots, $r_1 - r_2$ not integer] Find a series solution of $2xy'' + y' + xy = 0$.

$$y = x^r \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+r} \tag{5.4}$$

$$\begin{aligned} y' &= \sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} \\ &= [rc_0 x^{r-1} + (r+1)c_1 x^r + \cdots + (n+r)c_n x^{n+r-1} + \cdots] \\ &= x^{r-1}[rc_0 + (r+1)c_1 x + \cdots + (n+r)c_n x^n + \cdots] \end{aligned} \tag{5.5}$$

$$\begin{aligned} y'' &= \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} \\ &= [r(r-1)c_0 x^{r-2} + (r+1)rc_1 x^{r-1} + \cdots + (n+r)(n+r-1)c_n x^{n+r-2} + \cdots] \\ &= x^{r-2}[r(r-1)c_0 + (r+1)rc_1 x + \cdots + (n+r)(n+r-1)c_n x^n + \cdots] \end{aligned} \tag{5.6}$$

Subst. into the differential equation $2xy'' + y' + xy = 0$,

$$
\begin{aligned}
&2x^{r-1}[r(r-1)c_0 + (r+1)rc_1x + (r+2)(r+1)c_2x^2 + \cdots + (n+r)(n+r-1)c_nx^n + \cdots] \\
&+x^{r-1}[rc_0 + (r+1)c_1x + (r+2)c_2x^2 + \cdots + (n+r)c_nx^n + \cdots] \\
&+x^{r+1}[c_0 + c_1x + \cdots + c_nx^n + \cdots] = 0.
\end{aligned}
$$

Compare coefficients of x^{r-1}, x^r and x^{r+1}, we get :

$$[2r(r-1) + r]c_0 = 0 \tag{5.7}$$

$$[2(r+1)r + (r+1)]c_1 = 0 \tag{5.8}$$

$$2(n+r)(n+r-1)c_n + (n+r)c_n + c_{n-2} = 0, \; n \geq 2. \tag{5.9}$$

So we obtain $r = 0, \frac{1}{2}$. The equation $F(r) = r(2r-1) = 0$ is called the **indicial equation**. In this case we have

Distinct roots, $r_1 - r_2$ not integer

- Coeff. x^r: $[2(r+1)r + (r+1)]c_1 = (2r+1)(r+1)c_1 = 0 \Rightarrow c_1 = 0$.
- Coeff. x^{n+r-1}: $2(n+r)(n+r-1)c_n + (n+r)c_n + c_{n-2} = 0$. $(n \geq 2)$

Hence

$$c_n = \frac{-c_{n-2}}{(n+r)(2n+2r-1)}, \quad n \geq 2.$$

$c_1 = c_3 = c_5 = \cdots = 0$.

Two solutions are as follows:

Now we present more general case: Let $p(x) = \sum_{n=0}^{\infty} a_nx^n$, $q(x) = \sum_{n=0}^{\infty} b_nx^n$ and consider

$$x^2y'' + xp(x)y' + q(x)y = 0. \tag{5.10}$$

The derivatives of y are

$$
\begin{aligned}
y' &= \sum_{n=0}^{\infty}(n+r)c_nx^{n+r-1} \\
&= [rc_0x^{r-1} + (r+1)c_1x^r + \cdots + (n+r)c_nx^{n+r-1} + \cdots] \\
&= x^{r-1}[rc_0 + (r+1)c_1x + \cdots] \\
y'' &= \sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r-2} \\
&= [r(r-1)c_0x^{r-2} + (r+1)rc_1x^{r-1} + \cdots + (n+r)(n+r-1)c_nx^{n+r-2} + \cdots] \\
&= x^{r-2}[r(r-1)c_0 + (r+1)rc_1x + \cdots].
\end{aligned}
$$

Subst. these into (5.2) together with $p(x), q(x)$, and divide by x^r to obtain

$$[r(r-1)c_0 + (r+1)rc_1x + \cdots] + [a_0 + a_1x + \cdots][rc_0 + (r+1)c_1x + \cdots]$$
$$+[b_0 + b_1x + \cdots][c_0 + c_1x + \cdots] = 0.$$

Comparing coefficient, we see

$$[r(r-1) + a_0r + b_0]c_0 = 0.$$

Here c_0 is arb. Hence we obtain the following **indicial equation**.

$$F(r) := r(r-1) + a_0r + b_0 = 0$$

Denote the zeros by r_1, r_2. Coefficients of x^{r+n} :

$$F(r+n)c_n + \sum_{k=0}^{n} c_k[(r+k)a_{n-k} + b_{n-k}] = 0, \quad n \geq 1 \tag{5.11}$$

Example 5.2.4. [multiple roots, or $r_1 - r_2$ is an integer] The solution is complicated.

Theorem 5.2.5. *Assume the coefficients of the DE.*

$$x^2y'' + xp(x)y' + q(x)y = 0 \tag{5.12}$$

have power series

$$p(x) = \sum_{n=0}^{\infty} a_nx^n, \quad q(x) = \sum_{n=0}^{\infty} b_nx^n$$

convergent for $|x| < \rho$ and the roots of indical equation are $r_1, r_2 (r_1 \geq r_2)$. Then the equation (5.12) has the following type of solution which converges on $|x| < \rho$.

(1) r_1, r_2 are distinct and $r_1 - r_2$ not integer : There exists always two linearly independent solution of the form

$$y_1(x) = |x|^{r_1}(1 + c_1(r_1)x + c_2(r_1)x^2 + \cdots)$$

Here $c_n(r_1)$ is given by (5.11) with $(c_0 = 1, r = r_1)$.

$$y_2(x) = |x|^{r_2}(1 + c_1(r_2)x + c_2(r_2)x^2 + \cdots)$$

Here $c_n(r_2)$ is given by (5.11) with $(c_0 = 1, r = r_2)$.

(2) $r_1 - r_2 = N$ *integer:*

$$\begin{aligned} y_1(x) &= |x|^{r_1}(c_0 + c_1 x + \cdots) \\ y_2(x) &= C y_1(x)\ln x + |x|^{r_2}(b_0 + b_1 x + b_2 x^2 + \cdots). \end{aligned}$$

Here $c_0, c_1, b_0, b_1, \cdots$ *are given by (5.12) and* C *may be zero. If* $C = 0$ *then the two solutions are*

$$y_1(x) = |x|^{r_1}(c_0 + c_1 x + \cdots), \quad y_2(x) = |x|^{r_2}(b_0 + b_1 x + b_2 x^2 + \cdots).$$

(3) $r_1 = r_2$ *: The following type always solution exists.*

$$\begin{aligned} y_1(x) &= |x|^{r}(c_0 + c_1 x + \cdots) \\ y_2(x) &= y_1(x)\ln x + |x|^{r}(b_0 + b_1 x + b_2 x^2 + \cdots) \end{aligned}$$

This is a special case of (2) with $C = 1$ *(the logarithmic term always exists!).*

How to find the second solution?

With one solution $y_1(x)$ known in the above, you may try $y_2(x) = u(x)y_1(x)$ for the second solution. You will get

$$y_2(x) = y_1(x)\int \frac{e^{-\int^x P(t)dt}}{y_1^2(x)}dx. \tag{5.13}$$

5.3 Special Functions

5.3.1 Bessel Functions

The following DE is called the **Bessel's equation of order** ν.

$$\boxed{x^2 y'' + x y' + (x^2 - \nu^2)y = 0, \quad \nu \geq 0} \tag{5.14}$$

This equation arises in the study of heat equation or wave equation in cylindrical coordinates. Substituting $y = x^r \sum_{n=0}^{\infty} c_n x^n$ into the left of (5.14)

$$\begin{aligned} & \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r+2} - \nu^2 \sum_{n=0}^{\infty} c_n x^{n+r} \\ = & \sum_{n=0}^{\infty}\left\{(n+r)(n+r-1)c_n + (n+r)c_n - \nu^2 c_n\right\} x^{n+r} + \sum_{n=2}^{\infty} c_{n-2} x^{n+r} \\ = & \left\{r(r-1) + r - \nu^2\right\} c_0 x^r + \left\{(r+1)r + (r+1) - \nu^2\right\} c_1 x^{r+1} \\ & + \sum_{n=2}^{\infty}\left\{[(n+r)(n+r-1) + (n+r) - \nu^2]c_n + c_{n-2}\right\} x^{n+r} \end{aligned}$$

Compare coefficients of lowest degree terms,

$$\begin{aligned}
x^r &: (r^2-\nu^2)c_0 = 0 \\
x^{r+1} &: [(r+1)^2-\nu^2]c_1 = 0 \\
x^{r+n} &: [(n+r)^2-\nu^2]c_n + c_{n-2} = 0.
\end{aligned}$$

Indicial equation is $F(r) = r^2 - \nu^2 = 0$, so

$$r = \pm\nu.$$

From the coeff of x^{r+1} (choose $\nu \geq 0$ first)

$$((\pm\nu+1)^2 - \nu^2)c_1 = (2\nu+1)c_1 = 0, \tag{5.15}$$

we get $c_1 = 0$. From the coeff of x^{r+n} we get

$$[(n+r)^2-\nu^2]c_n + c_{n-2} = 0. \tag{5.16}$$

Hence $c_1 = c_3 = c_5 = \cdots = 0$. First consider $r = \nu$. It suffices to consider even terms, so let $n = 2k$. Then from (5.16) we see

$$c_{2k} = -\frac{c_{2k-2}}{2^2k(k+\nu)}.$$

Hence

$$\begin{aligned}
c_2 &= -\frac{c_0}{2^2 1\cdot(\nu+1)} \\
c_4 &= -\frac{c_2}{2^2\cdot 2(\nu+2)} = \frac{c_0}{2^4(1\cdot 2)(\nu+1)(\nu+2)} \\
c_6 &= -\frac{c_2}{2^2\cdot 3(\nu+3)} = \frac{-c_0}{2^6(1\cdot 2\cdot 3)(\nu+1)(\nu+2)(\nu+3)} \\
&\cdots \\
c_{2k} &= \frac{(-1)^k c_0}{2^{2k}k!(\nu+1)(\nu+2)\cdots(\nu+k)}.
\end{aligned}$$

Use a Gamma function defined by

$$\Gamma(x+1) = x\Gamma(x), \quad \Gamma(1) = 1.$$

We can easily see the following relation holds :

$$\begin{aligned}
\Gamma(\nu+k+1) &= (\nu+k)\Gamma(\nu+k) \\
&= (\nu+k)(\nu+k-1)\cdots(\nu+1)\Gamma(\nu+1).
\end{aligned}$$

If k is positive integer it holds that

$$\Gamma(k+1) = k!.$$

Since c_0 is arb. we let

$$c_0 = \frac{1}{2^\nu \Gamma(\nu+1)}$$

so that we have

$$c_{2k} = \frac{(-1)^k}{2^{2k+\nu} k! \Gamma(\nu+k+1)}.$$

The solution $y = \sum_{n=0}^{\infty} c_{2n} x^{2n+\nu}$ can be written as

$$J_\nu(x) = x^\nu \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k+\nu} k! \Gamma(\nu+k+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{2k+\nu}.$$

When $r = -\nu$, the solution is

$$J_{-\nu}(x) = x^{-\nu} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k-\nu} k! \Gamma(-\nu+k+1)} \left(\frac{x}{2}\right)^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(-\nu+k+1)} \left(\frac{x}{2}\right)^{2k-\nu}.$$

These $J_\nu, J_{-\nu}$ are called **Bessel's function of the first kind** of order ν and $-\nu$.

Remark 5.3.1. (1) If $\nu = 0$ these two functions are the same.

(2) If $\nu > 0$ and the difference $\nu - (-\nu) = 2\nu$ is not a positive integer then by case I above, $J_\nu, J_{-\nu}$ are linearly independent and the gen. solution is

$$y(x) = c_1 J_\nu(x) + c_2 J_{-\nu}(x).$$

(3) The case when ν is a half of odd integer, $\nu = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots$, $\nu - (-\nu) = 2\nu$ is an odd integer. In this case the two solutions are still linearly independent because the first terms of two solutions are x^ν, $x^{-\nu}$ resp.

(4) The case when ν is an integer, then $J_{-\nu}(x) = (-1)^\nu J_\nu(x)$.

Example 5.3.2. The gen. solution of the Bessel's equation $x^2 y'' + xy' + (x^2 - \frac{1}{4})y = 0$ is

$$y(x) = c_1 J_{\frac{1}{2}}(x) + c_2 J_{-\frac{1}{2}}(x).$$

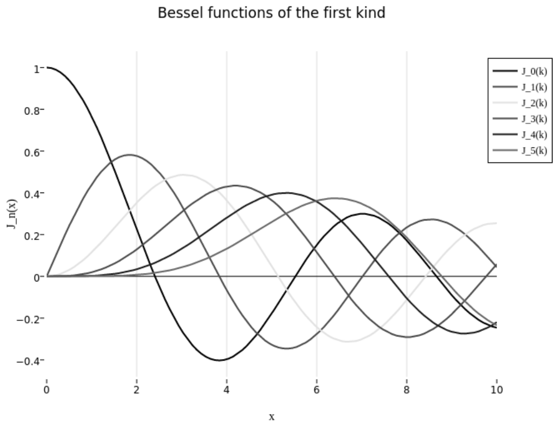

Figure 5.1: Bessel function of the first kind for $J_0, J_1, J_2, \cdots$

Bessel function of the second kind

If ν is not an integer, the function

$$Y_\nu(x) = \frac{1}{\sin \nu x}[J_\nu(x) \cos \nu\pi - J_{-\nu}(x)] \tag{5.17}$$

is a linearly independent solution of Bessel's equation. Hence the general solution is given by

$$y(x) = c_1 J_\nu(x) + c_2 Y_\nu(x).$$

Surprisingly this form of general solution also work when ν is an integer. Define for integer m,

$$Y_m(x) = \lim_{\nu \to m} Y_\nu(x). \tag{5.18}$$

Y_ν is called the **Bessel function of the second kind** of order ν.

A summary for Bessel equation

For any value ν the general solution of the Bessel equation

$$\boxed{x^2 y'' + xy' + (x^2 - \nu^2)y = 0, \quad \nu \geq 0} \tag{5.19}$$

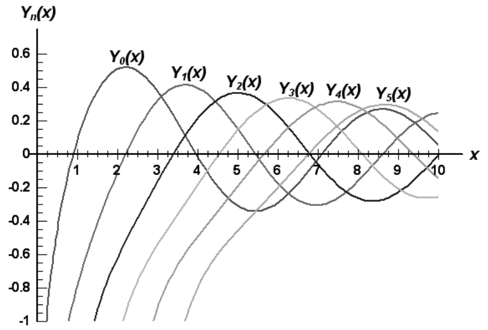

Figure 5.2: Bessel function of the second kind for $n = 0, 1, 2, \cdots$

is given by

$$y \;=\; c_1 J_\nu(x) + c_2 Y_\nu(x). \tag{5.20}$$

Example 5.3.3. The gen. solution of the Bessel's equation $x^2y'' + xy' + (x^2 - 16)y = 0$ is

$$y(x) = c_1 J_4(x) + c_2 Y_4(x).$$

DE. that can be solved in terms of Bessel functions

Consider the following DE:

$$x^2y'' + xy' + (\alpha^2x^2 - \nu^2)y = 0, \quad \nu > 0. \tag{5.21}$$

By change of variable $t = \alpha x, \alpha > 0$, we see

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \alpha\frac{dy}{dt}, \quad \frac{d^2y}{dx^2} = \frac{d}{dt}\frac{dy}{dx}\frac{dt}{dx} = \alpha^2\frac{d^2y}{dt^2}.$$

Thus the Bessel equation becomes

$$\left(\frac{t}{\alpha}\right)^2\alpha^2\frac{d^2y}{dt^2} + \left(\frac{t}{\alpha}\right)\alpha\frac{dy}{dt} + (t^2 - \nu^2)y = 0 \Rightarrow t^2y'' + ty' + (t^2 - \nu^2)y = 0, \quad \nu > 0. \tag{5.22}$$

The solution is now known as

$$y(t) = c_1 J_\nu(t) + c_2 Y_\nu(t).$$

Substitution $t = \alpha x$ gives

$$y(x) = c_1 J_\nu(\alpha x) + c_2 Y_\nu(\alpha x).$$

This equation is called the **parametric Bessel equation of order ν**.

Theorem 5.3.4. *We have the following*

1. *For $m = 0, 1, 2, \cdots$, $J_{-m}(x) = (-1)^m J_m(x)$.*
2. *$J_m(-x) = (-1)^m J_m(x)$.*
3. *$J_m(0) = 0$ if $m > 0$ and $J_0(0) = 1$.*
4. *$\lim_{x\to 0^+} Y_m(x) = -\infty$.*

5.3.2 Legendre Equation

The following type of DE. is called the **Legendre's equation**.

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0, \quad n \text{ real} \tag{5.23}$$

The solution of this equation is called the **Legendre function**. Let

$$y = \sum_{k=0}^{\infty} c_k x^k \tag{5.24}$$

and substitute into (5.23). With $\alpha = n(n+1)$ we have

$$\begin{aligned}
&(1 - x^2)\sum_{k=2}^{\infty} k(k-1)c_k x^{k-2} - 2x\sum_{k=1}^{\infty} kc_k x^{k-1} + \alpha\sum_{k=0}^{\infty} c_k x^k \\
&= \sum_{k=2}^{\infty} k(k-1)c_k x^{k-2} - \sum_{k=2}^{\infty} k(k-1)c_k x^k - 2\sum_{k=1}^{\infty} kc_k x^k + \alpha\sum_{k=0}^{\infty} c_k x^k = 0
\end{aligned}$$

and with shift of index we have

$$\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{k=2}^{\infty} k(k-1)c_k x^k - 2\sum_{k=1}^{\infty} kc_k x^k + \alpha\sum_{k=0}^{\infty} c_k x^k = 0.$$

Now

(1) Coeff. of 1 : $2c_2 + n(n+1)c_0 = 0$

(2) Coeff. of x : $6c_3 + [-2 + n(n+1)]c_1 = 0$

(3) Coeff. of x^k :

$$(k+2)(k+1)c_{k+2} + [-k(k-1) - 2k + n(n+1)]c_k = 0.$$

Thus

$$c_{k+2} = -\frac{(n-k)(n+k+1)}{(k+2)(k+1)}c_k, \quad k = 0, 1, \cdots, \tag{5.25}$$

where c_0, c_1 are arbitrary. For $k = 0, 1, 2, \cdots$ we see

$$\begin{aligned}
c_2 &= -\frac{n(n+1)}{2!}c_0 \\
c_3 &= -\frac{(n-1)(n+2)}{3!}c_1 \\
c_4 &= -\frac{(n-2)(n+3)}{4 \cdot 3}c_2 &&= \frac{(n-2)n(n+1)(n+3)}{4!}c_0 \\
c_5 &= -\frac{(n-3)(n+4)}{5 \cdot 4}c_3 &&= \frac{(n-3)(n-1)(n+2)(n+4)}{5!}c_1 \\
&\cdots && \cdots
\end{aligned}$$

Set $c_0 = 1$ and collect even number terms

$$y_1(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 + \cdots \tag{5.26}$$

$a_1 = 1$ and collect odd number terms

$$y_2(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - \cdots \tag{5.27}$$

Here y_1, y_2 are independent and interval of convergence is $|x| < 1$. Thus the general solution of (5.23) is given by

$$y(x) = c_1 y_1(x) + c_1 y_2(x).$$

Note that if n is even integer then the series for y_1 terminates (hence becomes a polynomial) and if n is odd integer then the series for y_2 terminates.

Legendre Polynomials

A special case of Legendre function when n is a natural number: If $k = n$ in (5.25) then $c_{n+2} = 0, c_{n+4} = 0, c_{n+6} = 0, \cdots$. If n is even $y_2(x)$ is a polyn. of degree n and if n is odd then $y_1(x)$ is a polyn. of degree n. These are **Legendre polynomials**.

Chapter 10

System of Linear Differential Equations

10.1 Theory of Linear System

More generally, we consider the first order system of linear differential equation in n-unknowns given by

$$\begin{array}{rcl} x_1' & = & a_{11}(t)x_1 + \cdots a_{1n}(t)x_n + f_1(t) \\ x_2' & = & a_{21}(t)x_1 + \cdots a_{2n}(t)x_n + f_2(t) \\ \cdot & \cdot & \cdots \\ x_n' & = & a_{n1}(t)x_1 + \cdots a_{nn}(t)x_n + f_n(t) \end{array} \tag{10.1}$$

In matrix form (10.1) becomes

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}, \tag{10.2}$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \ \mathbf{x}' = \begin{pmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{pmatrix}, \ A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}, \ \mathbf{f} = \begin{pmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

Linear dependence/independence

Given a set of solution vectors

$$\mathbf{x}^{(1)} = \begin{pmatrix} x_{11} \\ \vdots \\ x_{n1} \end{pmatrix}, \mathbf{x}^{(2)} = \begin{pmatrix} x_{12} \\ \vdots \\ x_{n2} \end{pmatrix}, \cdots, \mathbf{x}^{(n)} = \begin{pmatrix} x_{1n} \\ \vdots \\ x_{nn} \end{pmatrix},$$

the **Wronskian** W is defined as

$$W(\mathbf{x}^{(1)}, \cdots, \mathbf{x}^{(n)}) = \begin{vmatrix} x_{11}(t) & \cdots & x_{1n}(t) \\ \vdots & \cdots & \vdots \\ x_{n1}(t) & \cdots & x_{nn}(t) \end{vmatrix}. \tag{10.3}$$

Theorem 10.1.1. *[Criterion for linear independence] If $\mathbf{x}^{(1)}, \cdots, \mathbf{x}^{(n)}$ are solutions of (**??**) then the set of solution vectors are linearly independent if and only if*

$$W(\mathbf{x}^{(1)}, \cdots, \mathbf{x}^{(n)}) \neq 0. \tag{10.4}$$

for every t in the interval.

Theorem 10.1.2. *[Superposition principle] If $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \cdots, \mathbf{x}^{(n)}$ are the solutions of (**??**) then for any constants $c_1, c_2, \cdots, c_n$ the linear combination $c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \cdots + c_n\mathbf{x}^{(n)}$ is also a solution of (**??**).*

Definition 10.1.3. Any set $\mathbf{x}^{(1)}, \cdots, \mathbf{x}^{(n)}$ of n linearly independent solution vectors is said to be **fundamental set of solutions** of (**??**).

Nonhomogeneous System

The general solution of (**??**) is given by

$$\mathbf{x} = \mathbf{x}_c + \mathbf{x}_p,$$

where $\mathbf{x}_c = c_1\mathbf{x}^{(1)} + \cdots + c_n\mathbf{x}^{(n)}$ is the general solution of associated homogeneous system.

10.2 Homogeneous Linear System with constant coefficients

We will see the solution is generally given in this form when the matrix A has constant coefficients.

Eigenvalues and Eigenvectors

Given $n \times n$ matrix A consider the DE

$$\mathbf{x}' = A\mathbf{x}. \tag{10.5}$$

For a vector $\mathbf{k} \in \mathbb{R}^n$ we assume

$$\boxed{\mathbf{x} = \mathbf{k}e^{rt}} \tag{10.6}$$

and substitute into (10.5) we obtain

$$r\mathbf{k}e^{rt} = A\mathbf{k}e^{rt}.$$

we obtain

$$\boxed{A\mathbf{k} = r\mathbf{k}}.$$

From this we get

$$det(A - rI) = 0. \tag{10.7}$$

10.2.1 Real and distinct

When the eigenvalues of A are real and distinct, then general solution is given by

$$\mathbf{x}(t) = c_1\mathbf{k}^{(1)}e^{r_1 t} + c_2\mathbf{k}^{(2)}e^{r_2 t} + \cdots + c_3\mathbf{k}^{(n)}e^{r_n t}.$$

Example 10.2.1. Find the general solution of

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \mathbf{x}.$$

The characteristic equation is

$$(A - rI)\mathbf{k} = \begin{pmatrix} 1-r & 1 & 2 \\ 1 & 2-r & 1 \\ 2 & 1 & 1-r \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = 0. \tag{10.8}$$

$$\begin{aligned} |A - rI| &= \begin{vmatrix} 1-r & 1 & 2 \\ 1 & 2-r & 1 \\ 2 & 1 & 1-r \end{vmatrix} \\ &= -r^3 + 4r^2 + r - 4 = -(r-4)(r-1)(r+1) = 0. \end{aligned}$$

So $r_1 = 4, r_2 = 1, r_3 = -1$.

(1) $r = 4$:

$$\begin{pmatrix} -3 & 1 & 2 \\ 1 & -2 & 1 \\ 2 & 1 & -3 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = 0. \tag{10.9}$$

$$\begin{aligned} -3k_1 \quad +k_2 \quad +2k_3 &= 0 \\ k_1 \quad -2k_2 \quad +k_3 &= 0 \\ 2k_1 \quad +k_2 \quad -3k_3 &= 0. \end{aligned}$$

Choose $k_3 = 1$ so that

$$\begin{aligned} -3k_1 \quad +k_2 &= -2 \\ k_1 \quad -2k_2 &= -1 \\ 2k_1 \quad +k_2 &= 3 \end{aligned}$$

from which we obtain $k_1 = 1, k_2 = 1$, i.e.,

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{4t}.$$

(2) $r = 1$:

$$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = 0. \tag{10.10}$$

$$\begin{aligned} k_2 \quad +2k_3 &= 0 \\ k_1 \quad +k_2 \quad +k_3 &= 0 \\ 2k_1 \quad +k_2 &= 0. \end{aligned}$$

Choose $k_1 = 1$ so that

$$\begin{aligned} k_2 \quad +2k_3 &= 0 \\ k_2 \quad +k_3 &= -1 \\ k_2 &= -2 \end{aligned}$$

from which $k_2 = -2, k_3 = 1$, i.e.,

$$\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{t}.$$

(3) $r = -1$:

$$\begin{pmatrix} 2 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 2 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = 0. \tag{10.11}$$

$$\begin{array}{rrrl} 2k_1 & +k_2 & +2k_3 & = 0 \\ k_1 & +3k_2 & +k_3 & = 0 \\ 2k_1 & +k_2 & +2k_3 & = 0. \end{array}$$

Choose $k_3 = 1$ then

$$\begin{array}{rrl} 2k_1 & +k_2 & = -2 \\ k_1 & +3k_2 & = -1 \\ 2k_1 & +k_2 & = -2 \end{array}$$

from which $k_1 = -1, k_2 = 0$, i.e.,

$$\mathbf{x}^{(3)} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-t}.$$

Hence the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{t} + c_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{-t}.$$

Remark 10.2.2. In this example A is symmetric, in which case it is known that there always exist n linearly independent vectors. So finding the solution is simple.

Phase portrait or Phase plane

Example 10.2.3.

$$\mathbf{x}' = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \mathbf{x}.$$

Sol. The characteristic equation is

$$|A - rI| = \begin{vmatrix} 2-r & 3 \\ 2 & 1-r \end{vmatrix} = (r+1)(r-4) = 0, \ r_1 = -1, r_2 = 4.$$

For $r = -1$ the eigenvector is $\mathbf{k}_1 = (1, -1)^T$. For $r = 4$ the eigenvector is $\mathbf{k}_2 = (3, 2)^T$. So the solution of DE. is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t}.$$

If we eliminate parameter t and get relation between x and y, (use various constants) then we get certain relations. For example, if $c_1 = 1, c_2 = 0$, we get $x(t) = e^{-t}, y(t) = -e^{-t}$, hence $y = -x$. If $c_1 = 0, c_2 = 1$, we get $x(t) = 3e^{4t}, y(t) = 2e^{4t}$ and hence $y = \frac{2}{3}x$. These solutions corresponds to the two blue lines.

10.2.2 Repeated eigenvalues of multiplicity m

Assume r is a repeated eigenvalue of multiplicity m. There are two cases:

- There exists m linearly independent eigenvectors. In this case, the m-independent solutions are given by

$$c_1\mathbf{k}^{(1)}e^{r_1 t} + \cdots + c_m\mathbf{k}^{(m)}e^{r_m t}$$

- There exists only one linearly independent eigenvector $\mathbf{k}^{(1)}$ corresponding to the eigenvalue r. In this case, the m-linearly independent solutions are given by (Solve the system in this order)

$$\begin{aligned}\mathbf{x}_1 &= \mathbf{k}^{(1)}e^{r_1 t}\\ \mathbf{x}_2 &= \mathbf{k}^{(1)}te^{r_1 t} + \mathbf{k}^{(2)}e^{r_1 t}\\ \mathbf{x}_2 &= \mathbf{k}^{(1)}\frac{t^2}{2!}e^{r_1 t} + \mathbf{k}^{(2)}te^{r_1 t} + \mathbf{k}^{(3)}e^{r_1 t}\\ &= \cdots\end{aligned}$$

  Vectors $\mathbf{k}^{(1)}, \mathbf{k}^{(2)}$ through $\mathbf{k}^{(m)}$ are obtained by substituting these expressions into the D.E.

Less than m - Linearly independent eigenvectors - Second solution

When r is a multiple eigenvalue of multiplicity 2 and if there is only one eigenvector corresponding to it then the first solution is given by as before,

$$\mathbf{x}^{(1)} = \mathbf{k}e^{rt}, \tag{10.12}$$

where $\mathbf{k}$ satisfies

$$(A - rI)\mathbf{k} = 0. \tag{10.13}$$

The second solution is

$$\mathbf{x}^{(2)} = \mathbf{k}te^{rt} + \mathbf{p}e^{rt}, \tag{10.14}$$

where the vector $\mathbf{p}$ can be found by

$$(A - rI)\mathbf{p} = \mathbf{k}. \tag{10.15}$$

The final solution is

$$\mathbf{x} = c_1\mathbf{k}e^{rt} + c_2(\mathbf{k}te^{rt} + \mathbf{p}e^{rt}).$$

Example 10.2.4. Find the general solution of

$$\mathbf{x}' = \begin{pmatrix} 3 & -1 \\ 1 & 5 \end{pmatrix} \mathbf{x}. \tag{10.16}$$

Sol. The characteristic equation is

$$\begin{pmatrix} 3-r & -1 \\ 1 & 5-r \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{10.17}$$

$$|A - rI| = \begin{vmatrix} 3-r & -1 \\ 1 & 5-r \end{vmatrix} = (r-4)^2 = 0.$$

So $r = r_1 = r_2 = 4$ and the equation to for the eigenvectors is:

$$\begin{aligned} -k_1 \quad -k_2 \quad &= 0 \\ k_1 \quad +k_2 \quad &= 0. \end{aligned}$$

Solving it, we get $k_1 = 1, k_2 = -1$. Hence we have only one linearly independent vector:

$$\mathbf{k} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

from which we get one solution:

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t}.$$

We need to find another linearly independent solution. Recall scalar case, we tried: $x(t) = c_1 e^{rt} + c_2 t e^{rt}$. So we may try a solution like $\mathbf{k}te^{4t}$, but this is not enough! We have to add a term corresponding to the derivative of $\mathbf{k}te^{4t}$. Thus try

$$\boxed{\mathbf{x}^{(2)} = \mathbf{k}te^{4t} + \mathbf{p}e^{4t}}. \tag{10.18}$$

Substitute this into the DE., we get

$$(A - 4I)\mathbf{p} = \mathbf{k} \tag{10.19}$$

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{10.20}$$

So we obtain $p_1 + p_2 = -1$. Set $\eta_1 = k$ then $p_2 = -1 - k$ and we obtain

$$\mathbf{p} = \begin{pmatrix} k \\ -1-k \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + k \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Since the second term (in red) is absorbed into $\mathbf{k}$ (so into the first solution $\mathbf{x}^{(1)}$), we can set

$$\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{4t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{4t}.$$

So the general solution is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t} + c_2 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{4t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{4t} \right]$$

Example 10.2.5. Find the general solution of

$$\mathbf{x}' = \begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix} \mathbf{x}. \tag{10.21}$$

Sol. The characteristic equation is $(3-r)(-9-r)+36 = (r+3)^2 = 0$. The eigenvector are found from

$$\begin{pmatrix} 6 & -18 \\ 2 & -6 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{10.22}$$

We get one eigenvector $\mathbf{k} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. Hence $\mathbf{x}^{(1)} = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{-3t}$. For the second solution, we set

$$\mathbf{x}^{(2)} = \mathbf{k}te^{-3t} + \mathbf{p}e^{-3t}. \tag{10.23}$$

Substitute into DE., we see

$$(\mathbf{k}(1-3t) - 3\mathbf{p})e^{-3t} = (A\mathbf{k}t + A\mathbf{p})e^{-3t}.$$

Comparing, we get

$$(A+3I)\mathbf{k} = 0, \quad (A+3I)\mathbf{p} = \mathbf{k} = (3,1)^T.$$

$$(A+3I)\mathbf{p} = \mathbf{k} \Rightarrow \begin{pmatrix} 6 & -18 \\ 2 & -6 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}. \tag{10.24}$$

So $2p_1 - 6p_2 = 1$. We have has many solutions. Set p_2 free so that

$$\begin{pmatrix} 3p_2 + \frac{1}{2} \\ p_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} + p_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

As before, we can set $p_2 = 0$ to get $\mathbf{p} = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$, thus

$$\mathbf{x}^{(2)} = \mathbf{k}te^{-3t} + \mathbf{p}e^{-3t} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} te^{-3t} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} e^{-3t}.$$

Hence the final solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{-3t} + c_2 \left[\begin{pmatrix} 3 \\ 1 \end{pmatrix} te^{-3t} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} e^{-3t} \right].$$

10.2.3 Complex roots

Assume the characteristic equation of

$$\mathbf{x}' = A\mathbf{x} \tag{10.25}$$

has two complex conjugate roots $r_1 = \lambda + i\mu, r_2 = \lambda - i\mu$ with the corresponding eigenvectors $\mathbf{k}^{(1)}$ and $\mathbf{k}^{(2)}$. The solution in this case is

$$c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} = c_1\mathbf{k}^{(1)}e^{r_1 t} + c_2\mathbf{k}^{(2)}e^{r_2 t},$$

Since A is real, the eigenvectors corresponding to r_1, r_2 are two complex conjugates vectors $\mathbf{k}^{(1)}$ and $\mathbf{k}^{(2)} = \bar{\mathbf{k}}^{(1)}$. Set $\mathbf{k}^{(1)} = \mathbf{a} + i\mathbf{b}, \mathbf{k}^{(2)} = \mathbf{a} - i\mathbf{b}$.

we have

$$\begin{aligned} \mathbf{u} &= \tfrac{\mathbf{x}^{(1)}+\mathbf{x}^{(2)}}{2} &&= e^{\lambda t}(\mathbf{a}\cos\mu t - \mathbf{b}\sin\mu t) \\ \mathbf{v} &= \tfrac{\mathbf{x}^{(1)}-\mathbf{x}^{(2)}}{2i} &&= e^{\lambda t}(\mathbf{b}\cos\mu t + \mathbf{a}\sin\mu t) \end{aligned}$$

So we may write

$$\mathbf{x} = c_1\mathbf{u} + c_2\mathbf{v} = c_1 e^{\lambda t}(\mathbf{a}\cos\mu t - \mathbf{b}\sin\mu t) + c_2 e^{\lambda t}(\mathbf{a}\sin\mu t + \mathbf{b}\cos\mu t),$$

where $\mathbf{a}$ is the real part and $\mathbf{b}$ is the imaginary part of $\mathbf{k}^{(1)}$ respectively.

Example 10.2.6. Solve $\mathbf{x}' = \begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix} \mathbf{x}$.

Solution. The characteristic equation is

$$|A - rI| = \begin{vmatrix} 1-r & 3 \\ -3 & 1-r \end{vmatrix} = r^2 - 2r + 10 = 0$$

from which we obtain $r = 1 \pm 3i$. When $r_1 = 1 + 3i$

$$\begin{pmatrix} -3i & 3 \\ -3 & -3i \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{10.26}$$

We can choose eigenvectors

$$\mathbf{k}^{(1)} = \begin{pmatrix} 1 \\ i \end{pmatrix} \tag{10.27}$$

and the second vector is $\mathbf{k}^{(2)} = \overline{\mathbf{k}^{(1)}} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$. Hence

$$\mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(1+3i)t}, \quad \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{(1-3i)t}$$

or

$$\mathbf{u} = \frac{\mathbf{x}^{(1)} + \mathbf{x}^{(2)}}{2} = e^t \begin{pmatrix} \cos 3t \\ -\sin 3t \end{pmatrix}, \qquad \mathbf{v} = \frac{\mathbf{x}^{(1)} - \mathbf{x}^{(2)}}{2i} = e^t \begin{pmatrix} \sin 3t \\ \cos 3t \end{pmatrix}$$

Thus the general solution is

$$\mathbf{x}(t) = c_1 e^t \begin{pmatrix} \cos 3t \\ -\sin 3t \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin 3t \\ \cos 3t \end{pmatrix}$$

10.3 Diagonalization

10.4 Nonhomogeneous Linear Systems

We now study how to solve nonhomogeneous linear system of DE

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t). \tag{10.28}$$

As in the case of single DE. we separate the homogeneous case $\mathbf{x}' = A\mathbf{x}$ and the solution will be given by

$$\mathbf{x} = \mathbf{x}_h + \mathbf{x}_p,$$

where $\mathbf{x}_h$ is the solution of the homogeneous problem and $\mathbf{x}_p$ is a particular solution of the nonhomogeneous problem.

10.4.1 Method of Undetermined Coefficients

This works only when the coefficients of A are constant case, and right hand side terms are **constants, polynomials, exponential functions, sines, cosines or finite linear combinations of such functions**!

Example 10.4.1 (nonconstant rhs)**.** Solve $\mathbf{x}' = \begin{pmatrix} 6 & 1 \\ 4 & 3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 6t \\ -10t+4 \end{pmatrix}$.

Eigenvalues are $r_1 = 2, r_2 = 7$ and the eigenvectors are $\mathbf{x}_1 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Hence the complementary solution is

$$\mathbf{x}_c = c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{7t}.$$

For a **particular solution**, let

$$\mathbf{x}_p = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} t + \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$$

and substitute into the DE and find the numbers a_1, b_1, a_2, b_2.

$$\begin{aligned} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} &= \begin{pmatrix} 6 & 1 \\ 4 & 3 \end{pmatrix} \left[\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} t + \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \right] + \begin{pmatrix} 6 \\ -10 \end{pmatrix} t + \begin{pmatrix} 0 \\ 4 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} (6a_2 + b_2 + 6)t + 6a_1 + b_1 - a_2 \\ (4a_2 + 3b_2 - 10)t + 4a_1 + 3b_1 - b_2 + 4 \end{pmatrix} \end{aligned}$$

Hence

$$\begin{pmatrix} 6a_2 + b_2 + 6 & = & 0 \\ 4a_2 + 3b_2 - 10 & = & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 6a_1 + b_1 - a_2 & = & 0 \\ 4a_1 + 3b_1 - b_2 + 4 & = & 0 \end{pmatrix}$$

Solving first set of eqs we get $a_2 = -2, b_2 = 6$. We then substitute it into the second set of eqs to get $a_1 = -\frac{4}{7}, b_1 = \frac{10}{7}$. Therefore

$$\mathbf{x}_p = \begin{pmatrix} -2 \\ 6 \end{pmatrix} t + \begin{pmatrix} -\frac{4}{7} \\ \frac{10}{7} \end{pmatrix}.$$

and the general solution of DE is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{7t} + \begin{pmatrix} -2 \\ 6 \end{pmatrix} t + \begin{pmatrix} -\frac{4}{7} \\ \frac{10}{7} \end{pmatrix}.$$

10.4.2 Variation of Parameters

A Fundamental matrix - Homogeneous system

If $\mathbf{x}_1, \cdots, \mathbf{x}_n$ are fundamental set of solutions of homog. system $\mathbf{x}' = A\mathbf{x}$, then the general solution of homog. system is given by $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n$, or in matrix form

$$\mathbf{x} = \mathbf{\Phi}(t)\mathbf{c}, \tag{10.29}$$

where $\mathbf{c} = (c_1, c_2, \cdots, c_n)^T$, and $\mathbf{\Phi}(t)$ is the matrix whose columns are vectors $\mathbf{x}_i, i = 1, 2, \cdots, n$:

$$\mathbf{\Phi}(t) = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix}$$

called a **fundamental matrix**.

Variation of Parameters - Nonhomogeneous system

To find a particular solution we may try $\mathbf{x}_p = \mathbf{\Phi}(t)\mathbf{u}(t)$ and substitute into

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}. \tag{10.30}$$

Taking derivative we obtain

$$\mathbf{x}_p' = \mathbf{\Phi}(t)\mathbf{u}'(t) + \mathbf{\Phi}'(t)\mathbf{u}(t). \tag{10.31}$$

Substitute it into (10.30)

$$\mathbf{\Phi}(t)\mathbf{u}'(t) + \mathbf{\Phi}'(t)\mathbf{u}(t) = A\mathbf{\Phi}(t)\mathbf{u}(t) + \mathbf{f}(t). \tag{10.32}$$

Since $\mathbf{\Phi}'(t) = A\mathbf{\Phi}(t)$ we have

$$\mathbf{\Phi}(t)\mathbf{u}'(t) = \mathbf{f}(t). \tag{10.33}$$

$$\mathbf{u}'(t) = \mathbf{\Phi}(t)^{-1}\mathbf{f}(t) \;\Rightarrow\; \mathbf{u}(t) = \int \mathbf{\Phi}(t)^{-1}\mathbf{f}(t)dt.$$

Since $\mathbf{x}_p = \mathbf{\Phi}(t)\mathbf{u}(t)$ we have

$$\mathbf{x}_p(t) = \mathbf{\Phi}(t)\int \mathbf{\Phi}(t)^{-1}\mathbf{f}(t)dt. \tag{10.34}$$

Example 10.4.2. Solve the DE.

$$\mathbf{x} = \begin{pmatrix} -3 & 1 \\ 2 & -4 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 3t \\ e^{-t} \end{pmatrix}. \tag{10.35}$$

Eigenvectors corresponding to $r = -2, r = -5$ are

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

The solution of homog. system is

$$c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-2t} + c_2\begin{pmatrix} 1 \\ -2 \end{pmatrix}e^{-5t}.$$

The fundamental matrix is

$$\mathbf{\Phi}(t) = \begin{pmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{pmatrix} \text{ and } \mathbf{\Phi}(t)^{-1} = \begin{pmatrix} \frac{2}{3}e^{2t} & \frac{1}{3}e^{2t} \\ \frac{1}{3}e^{5t} & -\frac{1}{3}e^{5t} \end{pmatrix}.$$

Hence by (10.34)

$$\mathbf{x}_p(t) = \mathbf{\Phi}(t)\int \mathbf{\Phi}(t)^{-1}\mathbf{f}(t) = \begin{pmatrix} e^{-2t} & e^{-5t} \\ e^{-2t} & -2e^{-5t} \end{pmatrix}\int \begin{pmatrix} \frac{2}{3}e^{2t} & \frac{1}{3}e^{2t} \\ \frac{1}{3}e^{5t} & -\frac{1}{3}e^{5t} \end{pmatrix}\begin{pmatrix} 3t \\ e^{-t} \end{pmatrix} d$$

Hence the solution of the nonhomg system is

$$\mathbf{x}(t) = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-2t} + c_2\begin{pmatrix} 1 \\ -2 \end{pmatrix}e^{-5t} + \begin{pmatrix} \frac{6}{5}t - \frac{27}{50} + \frac{1}{4}e^{-t} \\ \frac{3}{5}t - \frac{21}{50} + \frac{1}{2}e^{-t} \end{pmatrix}$$

Initial Value Problems

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{c} + \mathbf{\Phi}(t)\int_{t_0}^{t} \mathbf{\Phi}(s)^{-1}\mathbf{f}(s)ds. \tag{10.36}$$

If the solution is to satisfy IC $\mathbf{x}(t_0) = \mathbf{x}_0$ then we must have $\mathbf{x}(t_0) = \mathbf{\Phi}(t_0)\mathbf{c}$, so

$$\mathbf{c} = \mathbf{\Phi}(t_0)^{-1}\mathbf{x}(t_0).$$

Hence the solution of IVP is

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{\Phi}(t_0)^{-1}\mathbf{x}(t_0) + \mathbf{\Phi}(t)\int_{t_0}^{t} \mathbf{\Phi}(s)^{-1}\mathbf{f}(s)ds. \tag{10.37}$$

10.4.3 Nonhomogeneous Problem by Diagonalization

Derivatives of e^{At}

The derivatives of a matrix function can be computed as

$$\frac{d}{dt}e^{At} = Ae^{At}. \tag{10.38}$$

e^{At} is a fundamental matrix

Any solution of homog. system $\mathbf{x}' = A\mathbf{x}$ is given by $e^{At}\mathbf{C}$ for some vector $\mathbf{C}$.

Nonhomog. systems

In view of techniques studied for scalar equations we can see the solution of $\mathbf{x}' = A\mathbf{x} + \mathbf{F}(t)$ is given by

$$\mathbf{x} = \mathbf{x}_c + \mathbf{x}_p = e^{At}\mathbf{C} + e^{At}\int_{t_0}^{t} e^{-As}\mathbf{F}(s)ds. \tag{10.39}$$

Laplace transform

Let us recall $\mathbf{X}(t) = e^{At}$ is the fundamental set of sols. satisfying the IC, i.e.

$$\mathbf{X}' = A\mathbf{X},\ \mathbf{X}(0) = I. \tag{10.40}$$

Use Laplace transform. If $\mathbf{x}(s) = \mathcal{L}\{\mathbf{X}(t)\} = \mathcal{L}\{e^{At}\}$, then we see

$$s\mathbf{x}(s) - \mathbf{X}(0) = A\mathbf{x}(s) \text{ or } (sI - A)\mathbf{x}(s) = I.$$

We have used small capital for transformed function and large capital for original function. Multiplying its inverse, we see

$$\mathbf{x}(s) = (sI - A)^{-1} I = (sI - A)^{-1}.$$

In other words, $\mathcal{L}\{e^{At}\} = (sI - A)^{-1}$ or

$$e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}. \tag{10.41}$$

Compare this with the formula:

$$e^{at} = \mathcal{L}^{-1}\{\frac{1}{(s-a)}\}.$$

This result can be used to find a matrix exponential.

Example 10.4.3. Use Laplace Transform to find e^{At} when

$$A = \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}. \tag{10.42}$$

In general a direction evaluation of e^{At} is very complicated. However, if we use Laplace Transform of e^{At} and do some algebraic manipulation on s-space, then use inverse Laplace Transform, we sometimes compute e^{At} easily.

Sol. First recall $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$ and so

$$\mathcal{L}\{e^{At}\} = (sI - A)^{-1} \text{ or } e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}. \tag{10.43}$$

We will compute $(sI - A)^{-1}$ first. Since

$$sI - A = \begin{pmatrix} s-1 & 1 \\ -2 & s+2 \end{pmatrix},$$

we have

$$(sI - A)^{-1} = \begin{pmatrix} s-1 & 1 \\ -2 & s+2 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{s+2}{s(s+1)} & \frac{-1}{s(s+1)} \\ \frac{2}{s(s+1)} & \frac{s-1}{s(s+1)} \end{pmatrix}.$$

Decomposing the entries we see

$$(sI - A)^{-1} = \begin{pmatrix} \frac{2}{s} - \frac{1}{s+1} & -\frac{1}{s} + \frac{1}{s+1} \\ \frac{2}{s} - \frac{2}{s+1} & -\frac{1}{s} + \frac{2}{s+1} \end{pmatrix}.$$

Taking the inverse Laplace Transform, we get by (10.43)

$$e^{At} = \begin{pmatrix} 2 - e^{-t} & -1 + e^{-t} \\ 2 - 2e^{-t} & -1 + 2e^{-t} \end{pmatrix}.$$

□

Chapter 11

System of Nonlinear Diff. Equation

11.1 Autonomous System, critical points, stability

Autonomous System

The DE. of the form

$$\begin{array}{rcl} \frac{dx_1}{dt} & = & g_1(x_1, x_2, \cdots, x_n) \\ \frac{dx_2}{dt} & = & g_2(x_1, x_2, \cdots, x_n) \\ \vdots & & \vdots \\ \frac{dx_n}{dt} & = & g_n(x_1, x_2, \cdots, x_n) \end{array} \tag{11.1}$$

is called **autonomous**. Notice that the equation does not have t explicitly.

Second Order DE as a System

A second order autonomous DE can be written as a system of first order autonomous DE. For example, given

$$x'' = F(x, x'), \tag{11.2}$$

we let $\frac{dx}{dt} = y$. Then $y' = x''$ and hence

$$\begin{array}{rcl} \frac{dx}{dt} & = & y \\ \frac{dy}{dt} & = & F(x, y). \end{array}$$

This is a system of first order autonomous system in x, y.

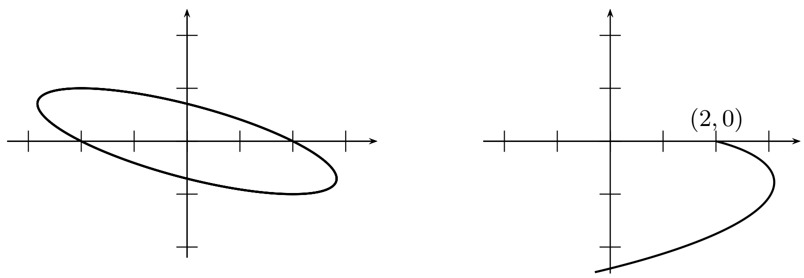

Figure 11.1: Example 11.1.4

Matrix form of autonomous system

If we use the vector (matrix) notation, we have $\mathbf{X}'(t) = \mathbf{g}(\mathbf{X})$ where

$$\mathbf{X}'(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad \mathbf{g}(\mathbf{X}) = \begin{pmatrix} g_1(x_1, \cdots, x_n) \\ \vdots \\ g_n(x_1, \cdots, x_n) \end{pmatrix}.$$

Example 11.1.1 (Periodic solutions-Check the Fig. 11.1)**.** Solve

$$(1) \left\{ \begin{array}{rcl} x' & = & 2x + 8y \\ y' & = & -x - 2y \end{array} \right. \text{ and } (2) \left\{ \begin{array}{rcl} x' & = & x + 2y \\ y' & = & -\frac{1}{2}x + y \end{array} \right.$$

In each case sketch the graph when $\mathbf{x}(0) = (2, 0)$.

Sol. (1). In Section 10.2, we have seen the solution is

$$\begin{array}{rcl} x & = & c_1(2\cos 2t - 2\sin 2t) + c_2(2\cos 2t + 2\sin 2t) \\ y & = & c_1(-\cos 2t - 2) - c_2 \sin 2t. \end{array}$$

with IC, we get

$$x = 2\cos 2t + 2\sin 2t, \; y = -\sin 2t.$$

These are clearly periodic. We can eliminate t and get $(\frac{x+2y}{2})^2 + y^2 = 1$. Figure 11.1 (1)

Changing to Polar coordinates

$r^2 = x^2 + y^2, \; \theta = \tan^{-1} \frac{y}{x}, \; \frac{\partial r}{\partial x} = \frac{x}{r} \; \frac{\partial r}{\partial y} = \frac{y}{r}$.

$$\frac{\partial \theta}{\partial x} = \frac{-y}{x^2 + y^2} = \frac{-y}{r^2}, \; \frac{\partial \theta}{\partial x} = \frac{x}{x^2 + y^2} = \frac{x}{r^2}.$$

Example 11.1.2. Find the solution of

$$\begin{aligned} \frac{dr}{dt} &= 0.5(3-r) \\ \frac{d\theta}{dt} &= 1. \end{aligned} \tag{11.3}$$

with IC. $\mathbf{x}(0) = (0,1)$, and with IC. $\mathbf{x}(0) = (3,0)$.

Sol.

$$r = 3 + c_1 e^{-0.5t}, \ \theta = t + c_2.$$

With IC. $\mathbf{x}(0) = (0,1)$,

$$\begin{aligned} x(0) = (3 + c_1 e^{-0.5t})\cos(t + c_2) &= 0 \\ y(0) = (3 + c_1 e^{-0.5t})\sin(t + c_2) &= 1. \end{aligned}$$

$$\begin{aligned} x(0) = (3 + c_1)\cos(c_2) &= 0 \\ y(0) = (3 + c_1)\sin(c_2) &= 1. \end{aligned}$$

Hence we get $c_1 = -2, c_2 = \pi/2$. The solution is the spiral $r = 3 - 2e^{-0.5(\theta-\pi/2)}$. As $\theta \to \infty$ the path approaches a circle. Fig **??**

Next, with IC. $\mathbf{x}(0) = (3,0)$ $r = 3, \theta = 0$ when $t = 0$. Thus $c_1 = c_2 = 0$. Thus

$$r = 3, \ \theta = t \ \Rightarrow x = 3\cos t, y = 3\sin t.$$

□

11.2 Stability of Linear System

We again consider a plane autonomous DE.

$$\begin{aligned} \frac{dx}{dt} &= P(x,y) \\ \frac{dy}{dt} &= Q(x,y). \end{aligned} \tag{11.4}$$

Recall the definition of a critical points: $P(x,y) = Q(x,y) = 0$.

Stability Analysis

For the stability of the generally nonlinear system (11.4), we first study the stability of the linear system.

$$\mathbf{x}' = A\mathbf{x} \ \text{ or } \ \left\{ \begin{aligned} \frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy \end{aligned} \right. , \tag{11.5}$$

The behavior depends on the eigenvalues.

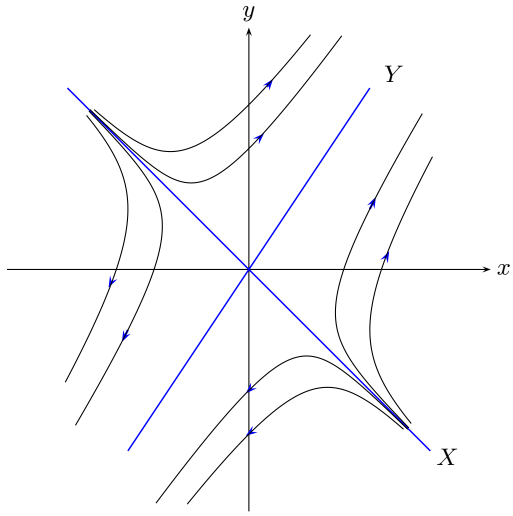

Figure 11.2: Example **??**, Saddle

Case I: Real and distinct eigenvalues; $\tau^2 - 4\Delta > 0$.

Consider

$$\begin{cases} \frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy. \end{cases} \tag{11.6}$$

The solution of (11.7) is given by the following form:

$$\mathbf{x}(t) = c_1 \boldsymbol{\xi} e^{\lambda_1 t} + c_2 \boldsymbol{\eta} e^{\lambda_2 t}. \tag{11.7}$$

This is again classified as follows:

- **Stable node** if both λ_1, λ_2 are negative ($\tau^2 - 4\Delta > 0$, $\tau < 0$, and $\Delta > 0$)
- **Unstable node** if both λ_1, λ_2 are positive ($\tau^2 - 4\Delta > 0$, $\tau > 0$, and $\Delta > 0$)
- **Saddle** if $\lambda_1 \lambda_2 < 0$ ($\tau^2 - 4\Delta > 0$, and $\Delta < 0$) Saddle is unstable.

Example 11.2.1 (**Real and distinct; different sign → Saddle).**

$$\begin{vmatrix} 1 - \lambda & 2 \\ 3 & 2 - \lambda \end{vmatrix} = \lambda^2 - 3\lambda - 4 = 0 \Rightarrow \lambda = 4, -1.$$

We have charact. equation

$$\lambda^2 - \tau\lambda + \Delta = 0.$$

$\Delta = ad - bc = -4$ and trace $\tau = a + d = 3$.

For $\lambda = 4$, $\boldsymbol{\xi} = (2,3)^T$, and for $\lambda = -1$, $\boldsymbol{\eta} = (1,-1)^T$. Thus the solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$$

(1) If $c_1 = 0$ we see $y = -x$.

(2) If $c_2 = 0$ we see $y = \frac{3}{2}x$.

(3) If $c_1 \neq 0, c_2 \neq 0$.

Treat $(1,-1)$ direction as if X-axis, $(2,3)$ direction as if Y-axis. Then we have $Y = \frac{c}{X^4}$, a hyperbola.

It is a **saddle**.

Example 11.2.2 (Real and distinct ; same sign → Nodes).

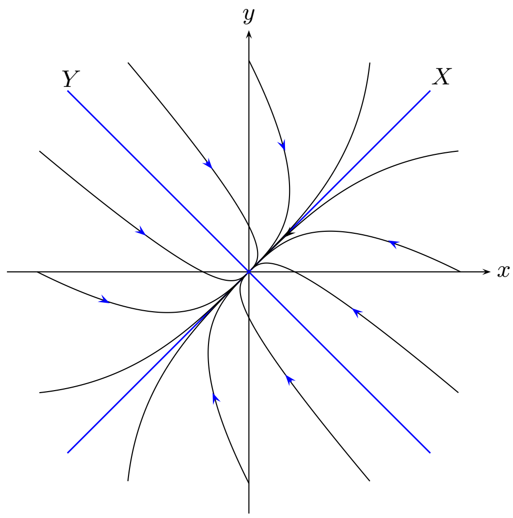

Figure 11.3: Example 11.2.2, Node

Sol. The critical point is $(0,0)$. Charac. eq. is

$$\begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 4\lambda + 3 = 0, \;\; \lambda = -1, -3.$$

and eigenvectors are

$$\xi_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t}.$$

Repeated real eigenvalues ($\tau^2 - 4\Delta = 0$; same sign $\to$ Nodes)

Degenerate nodes:

(1) Two linearly independent eigenvectors

$$\mathbf{X}(t) = c_1 \boldsymbol{\xi} e^{\lambda_1 t} + c_2 \boldsymbol{\eta} e^{\lambda_1 t}.$$

If $\lambda_1 < 0$ then it is stable, otherwise unstable.

(2) Single linearly independent eigenvector

$$\mathbf{X}(t) = c_1 \boldsymbol{\xi} e^{\lambda_1 t} + c_2 (\boldsymbol{\xi} t e^{\lambda_1 t} + \boldsymbol{\eta} e^{\lambda_1 t}),$$

where $(A - \lambda_1 I)\boldsymbol{\eta} = \boldsymbol{\xi}$. If $\lambda_1 < 0$ then it is stable. It can be written as

$$\mathbf{X}(t) = t e^{\lambda_1 t} \left[c_2 \boldsymbol{\xi} + \frac{c_1}{t} \boldsymbol{\xi} + \frac{c_2}{t} \boldsymbol{\eta} \right].$$

As $t \to \infty$ the solution approaches the direction of $\boldsymbol{\xi}$. (Only one direction.) So it is called **Degenerate stable node**.

Complex Eigenvalues ($\tau^2 - 4\Delta < 0$; $\to$ Spiral)

Example 11.2.3.

$$\begin{cases} x' &= \alpha x + \beta y \\ y' &= -\beta x + \alpha y \ (\alpha, \beta \text{ real}, \ \beta > 0) \end{cases} \tag{11.8}$$

Eigenvalues are $\alpha \pm i\beta$. Hence this is a **spiral**.

If $\alpha = 0$ we have a periodic solution. More generally, when the eigenvalues are $\lambda = \alpha \pm i\beta$ with corresponding eigenvectors $\mathbf{a}_1 \pm i\mathbf{a}_2$, then

$$\mathbf{x}_1(t) = (\mathbf{a}_1 \cos\beta t - \mathbf{a}_2 \sin\beta t)e^{\alpha t}, \quad \mathbf{x}_2(t) = (\mathbf{a}_2 \cos\beta t + \mathbf{a}_1 \sin\beta t)e^{\alpha t}.$$

So

$$x(t) = (c_{11} \cos\beta t + c_{12} \sin\beta t)e^{\alpha t}, \quad y(t) = (c_{21} \cos\beta t + c_{22} \sin\beta t)e^{\alpha t}.$$

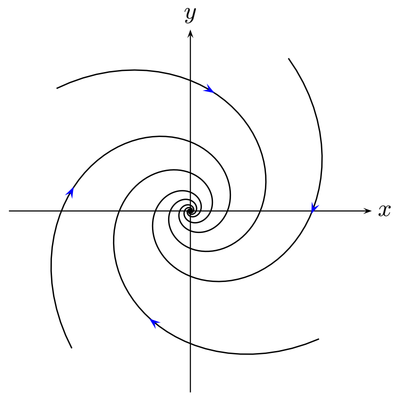

Figure 11.4: spiral

Stability: Linear case

Roots of Char. eq.	Critical point(Linear)	Stability (Linear)
$r_1 > r_2 > 0$	node	unstable
$r_1 < r_2 < 0$	node	stable, attr.
$r_1 \cdot r_2 < 0$	saddle	unstable
$r_1 = r_2 < 0$	node	stable, attr.
$r_1 = r_2 > 0$	node	unstable
$\alpha \pm i\beta, \alpha > 0$	spiral	unstable
$\alpha \pm i\beta, \alpha < 0$	spiral	stable, attr.
$\alpha = 0,\ \pm i\beta$	center	stable

Classification of Critical Points- Linear Case

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy. \end{aligned} \tag{11.9}$$

Its charact. equation is

$$\lambda^2 - (a+d)\lambda + ad - bc = \lambda^2 - \tau\lambda + \Delta = 0.$$

Let $\tau = a + d$, and the determinant Δ be $ad - bc$. We classify the critical points according to $\tau^2 - 4\Delta$:

(1) Real roots of same sign $\Delta > 0$, $\tau^2 - 4\Delta \geq 0$: node

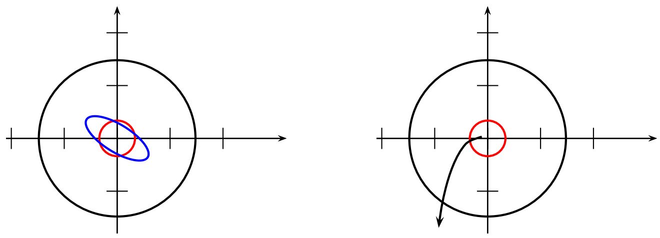

Figure 11.5: Stable, unstable critical points

(2) Real roots of opposite sign $\Delta < 0$: saddle

(3) Complex roots $\tau \neq 0, \tau^2 - 4\Delta < 0$: spiral

(4) Pure imaginary roots $\tau = 0, \quad \Delta > 0$: center

11.3 Nonlinear system-Linearization

An autonomous nonlinear system is given as

$$\begin{cases} \frac{dx}{dt} &= F(x,y) \\ \frac{dy}{dt} &= G(x,y). \end{cases} \tag{11.10}$$

The zeros of $F(x,y) = G(x,y) = 0$ are called **critical points**.

Definition 11.3.1 (stable critical point)**.** Let $\mathbf{x}_1$ be a critical point of a autonomous system. It is called a **stable critical point** if for any radius $\rho > 0$ there exists a radius $r > 0$ such that if the initial position satisfies $|\mathbf{x}_0 - \mathbf{x}_1| < r$, then the corresponding solution $\mathbf{x}(t)$ satisfies $|\mathbf{x}(t) - \mathbf{x}_1| < \rho$ for all $t > 0$. If, in addition, the solution satisfies $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}_1$ whenever $|\mathbf{x}_0 - \mathbf{x}_1| < r$ then $\mathbf{x}_l$ is called an **asymptotically stable critical point**.

Otherwise, a critical point $\mathbf{x}_1$ is called an **unstable critical point**. (see the book for more precise defintion)

Example 11.3.2.

$$\begin{aligned} \frac{dr}{dt} &= 0.05r(3-r) \\ \frac{d\theta}{dt} &= -1. \end{aligned} \tag{11.11}$$

Show that $(0,0)$ is an unstable critical point. Solving the system directly in terms of r and θ,

$$\frac{dr}{r(3-r)} = 0.05dt \Rightarrow \ln\frac{r}{3-r} = 0.15t + c \Rightarrow \frac{r}{3-r} = Ce^{0.15t}$$

$$r = \frac{3}{1 + c_0 e^{-0.15t}}.$$

With IC, $r(0) = r_0$, we get $c_0 = (3 - r_0)/r_0$. As $t \to \infty$ we see $r(t) \to 0$. $r = 3 - 2e^{-0.5(\theta - \pi/2)}$. As $\theta \to \infty$ the path approaches a circle of radius 3. Hence the circle is stable.(limit cycle)

Linearization

Consider the following nonlinear system of DE.

$$\begin{cases} x' &= P(x, y) \\ y' &= Q(x, y) \end{cases} \quad \text{or } \mathbf{x}' = \mathbf{g}(\mathbf{x}), \tag{11.12}$$

where $P(x, y), Q(x, y)$ are C^2-functions. Assume $\mathbf{x}_1 = (x_0, y_0)$ is a critical point. We linearize this using the Taylor expansion at (x_0, y_0).

The vector form of the system of equation is

$$\mathbf{x}' = \mathbf{g}(\mathbf{x}) = \mathbf{g}(\mathbf{x}_1) + A(\mathbf{x} - \mathbf{x}_1) + o(\|\mathbf{x}\|) \approx A(\mathbf{x} - \mathbf{x}_1),$$

where A is the Jacobian matrix

$$A = \begin{pmatrix} P_x & P_y \\ Q_x & Q_y \end{pmatrix} \Bigg|_{(x_0, y_0)}$$

The system (with $\mathbf{x}_1 = 0$) the system $\mathbf{x}' = A\mathbf{x}$ is called the **linearization** of (11.15).

Theorem 11.3.3. *Assume $\mathbf{x}_1$ is a critical point of the plane autonomous system $\mathbf{x}' = \mathbf{g}(\mathbf{x})$.*

(1) If the eigenvalues of $A = \mathbf{g}'(\mathbf{x}_1)$ has negative real part, then $\mathbf{x}_1$ is an asymptotically stable critical point.

(2) If $A = \mathbf{g}'(\mathbf{x}_1)$ has an eigenvalue with positive real part, then $\mathbf{x}_1$ is an unstable critical point.

Example 11.3.4. (a) Classify the critical points of

$$\begin{cases} x' &= x^2 + y^2 - 6 \\ y' &= x^2 - y. \end{cases}$$

The critical points are $(\sqrt{2}, 2)$ and $(-\sqrt{2}, 2)$.

$$\mathbf{g}'(\mathbf{x}) = \begin{pmatrix} 2x & 2y \\ 2x & -1 \end{pmatrix}$$

and

$$A_1 = \begin{pmatrix} 2\sqrt{2} & 4 \\ 2\sqrt{2} & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -2\sqrt{2} & 4 \\ -2\sqrt{2} & -1 \end{pmatrix}$$

(1) A_1. $\Delta = (-2\sqrt{2} - 8\sqrt{2}) < 0$, $\tau > 0$. So A_1 has a positive eigenvalue and a negative eigenvalue, so unstable(saddle).

(2) A_2. $\Delta = (2\sqrt{2} + 8\sqrt{2}) > 0$, $\tau < 0$. So both eigenvalues are negative real, so stable.

Nonlinear case - from linearization

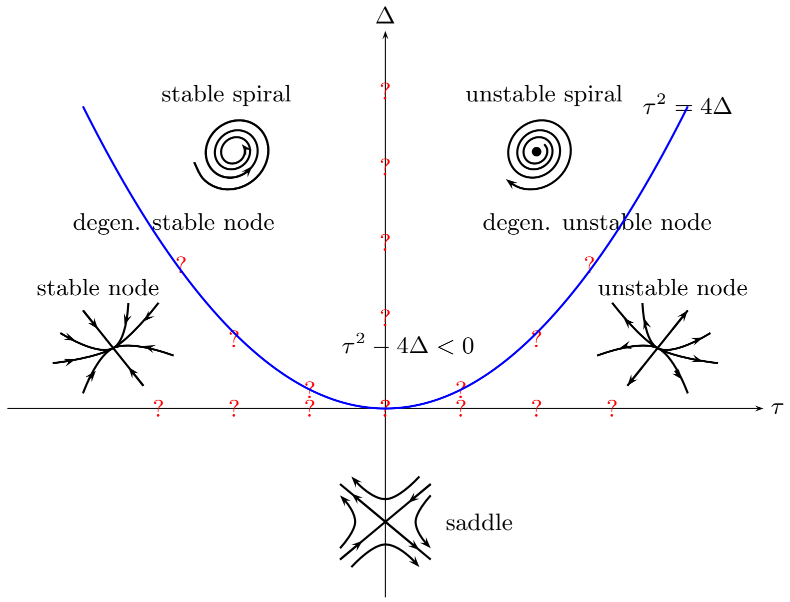

Figure 11.6: Classification- **nonlinear** case

Theorem 11.3.5. *Stability: nonlinear system*

char. value	*point (linear)*	*Stab.(linear)*	*point(nonlin)*	*Stab.(nonlin)*
$r_1 > r_2 > 0$	*node*	*unstable*	*node*	*unstable*
$r_1 < r_2 < 0$	*node*	*stable, attr.*	*node*	*stable, attr.*
$r_1 \cdot r_2 < 0$	*saddle*	*unstable*	*saddle*	*unstable*
$r_1 = r_2 < 0$	*node*	*stable, attr.*	*node*	*stable, attr.*
$r_1 = r_2 > 0$	*node*	*unstable*	*node*	*unstable*
$\alpha \pm i\beta, \alpha > 0$	*spiral*	*unstable*	*spiral*	*unstable*
$\alpha \pm i\beta, \alpha < 0$	*spiral*	*stable, attr.*	*spiral*	*stable, attr.*
$\alpha = 0, \pm i\beta$	*center*	*stable*	*center, spiral*	*indeterm.*

Sol. $(0,0)$ is a critical point. Differentiate F, G, we get $\mathbf{x}' = A\mathbf{x}$ where

$$A = DF(0,0) = \begin{pmatrix} 1+2x & 4y \\ 2+y & 1+x \end{pmatrix}_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

Example 11.3.6 (Soft Spring)**.** $mx'' + kx + k_1x^3 = 0$, $k = 1 > 0$, $k_1 = -1 < 0$. $m = 1$. By introducing $y = x'$, we obtain a system that can be written as

$$\begin{cases} x' &= y \\ y' &= x^3 - x. \end{cases}$$

Find and classify critical points.

Sol. We see the critical points are $(0,0)$, $(1,0)$ and $(-1,0)$. Differentiating g, we get $\mathbf{x}' = A\mathbf{x}$ where

$$A_1 = Dg(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A_2 = Dg(1,0) = Dg(-1,0) = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$$

The eigenvalues of A_1 are $\pm i$. So we are not sure about the stability. The eigenvalues of A_2 are $\pm\sqrt{2}$. So saddle.

The phase plane method

Example 11.3.7. Consider the D.E.

$$\begin{cases} x' &= y^2 \\ y' &= x^2. \end{cases}$$

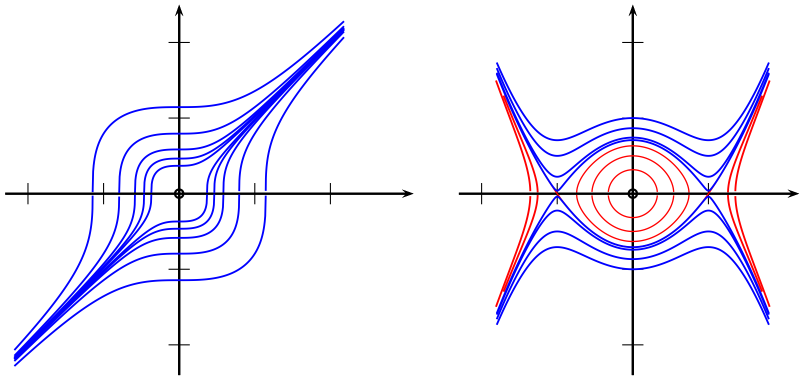

Figure 11.7: Phase Plane of Example 8, 9

The critical point is $(0,0)$. Since the determinant of linearization

$$\mathbf{g}' = \begin{pmatrix} 0 & 2y \\ 2x & 0 \end{pmatrix}$$

is zero (on the border line), the nature of the critical point is in doubt. Instead, solving

$$\frac{dy}{dx} = \frac{x^2}{y^2},$$

we get $y^3 = x^3 + c$ or $y = \sqrt[3]{x^3 + c}$. If $\mathbf{X}(0) = (0, y_0)$ then $y^3 = x^3 + y_0^3$. From the Figure 11.7, we conclude it is unstable.

Example 11.3.8 (Phase plane analysis of Soft Spring)**.** Investigate the behavior of critical point of

$$mx'' + x - x^3 = 0.$$

Use $x' = y$, $y' = (x^3 - x)/m$ to get

$$\begin{cases} x' &= y \\ y' &= \frac{x^3 - x}{m}. \end{cases}$$

Set $m = 1$. The critical point is $(0,0)$.

$$\frac{dy}{dx} = \frac{x^3 - x}{y}. \tag{11.13}$$

Separation of var.

$$\frac{y^2}{2} = \frac{x^4}{4} - \frac{x^2}{2} + c.$$

$$y^2 = \frac{(x^2-1)^2}{2} + c_0.$$

If $\mathbf{X}(0) = (x_0, 0)$ then

$$0 = \frac{(x_0^2-1)^2}{2} + c_0 \Rightarrow c_0 = -\frac{(x_0^2-1)^2}{2}.$$

So

$$\begin{aligned} y^2 &= \frac{(x^2-1)^2}{2} - \frac{(x_0^2-1)^2}{2} = \frac{1}{2}[(x^2-1)-(x_0^2-1)][(x^2-1)+(x_0^2-1)] \\ &= \frac{1}{2}(x^2-x_0^2)(x^2-2+x_0^2). \end{aligned}$$

Investigate near the point $(0,0)$. Set $y = 0$, we get $x = \pm x_0$ and the right hand side is positive only when $-x_0 < x < x_0$. The origin is center. Note that when $x_0 = 1$, $\sqrt{2}y^2 = (x^2-1)$. it is a quadratic poly. for $x \geq 1$. See Figure 11.7.

Remark 11.3.9. We only checked when the initial point x_0 is close to the origin.

11.4 Autonomous system as mathematical models

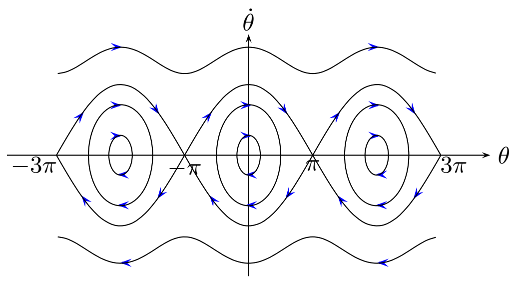

Figure 11.8: Phase plane of Pendulum

Example 11.4.1. [Nonlinear pendulum - no friction] Recall the movement of the Pendulum in section 1.

$$\theta'' + \frac{g}{\ell}\sin\theta = 0. \tag{11.14}$$

Let $x = \theta$, $y = x'$. Then the movement of the pendulum is described by

$$\begin{cases} x' &= y \\ y' &= -\frac{g}{\ell}\sin x. \end{cases}$$

Solution. The critical points are $(k\pi, 0), k = \pm 1, \pm 2, \cdots$.

(1) Let $k = (2n+1)$. Then the critical point is $((2n+1)\pi, 0)$. Linearizing at $((2n+1)\pi, 0)$,

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{g}{\ell}\cos x & 0 \end{pmatrix}_{((2n+1)\pi,0)} = \begin{pmatrix} 0 & 1 \\ \frac{g}{\ell} & 0 \end{pmatrix}.$$

Since $\Delta = -g/\ell < 0$ the eigenvalue values are distinct real. Hence critical points are saddle. Original nonlinear system is also saddle.

(2) Let $k = 2n$. Then the critical point is $(2n\pi, 0)$. Linearizing at $(2n\pi, 0)$,

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{g}{\ell}\cos x & 0 \end{pmatrix}_{(2n\pi,0)} = \begin{pmatrix} 0 & 1 \\ -\frac{g}{\ell} & 0 \end{pmatrix}.$$

The eigenvalues of A are pure imaginary, so it is a center. By Theorem 11.3.8 the original point is either center or spiral. However, the stability behavior of nonlinear system is doubtful. So try

$$\frac{dy}{dx} = -\frac{g}{\ell}\frac{\sin x}{y}.$$

Solving, we get

$$y^2 = \frac{2g}{\ell}\cos x + c.$$

With I.C $\mathbf{x}(0) = (x_0, 0)$, we have

$$y^2 = \frac{2g}{\ell}(\cos x - \cos x_0).$$

For this to have solutions, we need $\cos x - \cos x_0 \geq 0$. Near the origin, we require $|x| < x_0$. This sol is periodic.

Example 11.4.2. [Periodic solution of pendulum- with initial angular velocity] We assume the pendulum at $\theta = 0$ is given an initial angular velocity ω_0 rad/s. Determine under what condition the motion is periodic.

Sol. Use IC. $\mathbf{x}(0) = (0, \omega_0)$ to $y^2 = \frac{2g}{\ell}\cos x + c$ to get

$$\omega_0^2 = \frac{2g}{\ell} + c$$

Hence

$$y^2 = \frac{2g}{\ell}(\cos x - 1 + \frac{\ell}{2g}\omega_0^2).$$

Example 11.4.3. [Pendulum- with friction] We may assume the friction is proportional to the velocity, i.e., the friction is $c\ell\theta'$. Hence we have

$$m\ell^2\theta'' + c\ell\theta' + mg\ell\sin\theta = 0.$$

Dividing by $m\ell^2$

$$\theta'' + a\theta' + b\sin\theta = 0, \quad a = \frac{c}{m\ell}, \quad b = \frac{g}{\ell}.$$

Let $x = \theta, y = \theta'$. Then we get

$$\begin{cases} x' &= y \\ y' &= -b\sin x - ay. \end{cases} \tag{11.15}$$

Its critical points are $(n\pi, 0), n = \pm 1, \pm 2, \cdots$. Linearizing at $(0, 0)$ we get

$$\mathbf{x}' = A\mathbf{x} = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}\mathbf{x}.$$

The char. values are $r_1, r_2 = \frac{-a\pm\sqrt{a^2-4b}}{2}$. According to location of (a, b) we have

(1) $a^2 - 4b > 0$: char. values are distinct negative real: So the critical points are stable and node. Same for the original problem.

(2) $a^2 - 4b = 0$: char. values are double negative real:So the critical points are stable and node. The original has node or spiral(stable).

(3) $a^2 - 4b < 0$: char. values are complex with negative real part. So the critical points are stable. Same for the original problem.

At $(2m\pi, 0), m = 1, 2, 3, \cdots$ we can show the same behavior. Now look at $((2m-1)\pi, 0), m = \pm 1, \pm 2, \cdots$. Linearizing, we get $A = \begin{pmatrix} 0 & 1 \\ b & -a \end{pmatrix}$. In this case the char. values are $\frac{-a\pm\sqrt{a^2+4b}}{2}$. In this case, it is a saddle. Same for the original problem.

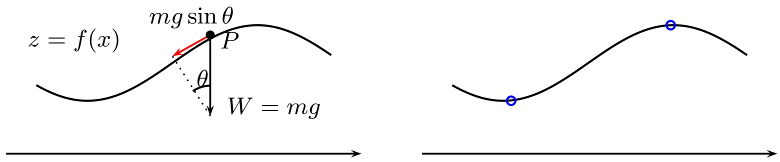

Figure 11.9: Sliding bead and critical points

Nonlinear Oscillation: Sliding bead

Suppose a bead is sliding along a wire forming a curve described by the function $z = f(x)$.

Example 11.4.4. [Sliding bead]

$$F_x = -mg\sin\theta\cos\theta = -mg\tan\theta\cos^2\theta = -mg\frac{f'(x)}{1+[f'(x)]^2}.$$

Assume a damping force $-\beta x'$ (proportional to velocity). Then the movement of the sliding bead is described by

$$mx'' = -mg\frac{f'(x)}{1+[f'(x)]^2} - \beta x'. \tag{11.16}$$

Hence we have

$$\begin{cases} x' &= y \\ y' &= -g\frac{f'(x)}{1+[f'(x)]^2} - \frac{\beta}{m}y. \end{cases}$$

The critical points $\mathbf{x}_1 = (x_1, y_1)$ satisfy $y_1 = 0, f'(x_1) = 0$ (local extreme point of $z = f(x)$). After some algebra, we can see

$$\mathbf{g}'(\mathbf{x}_1) = \begin{pmatrix} 0 & 1 \\ -gf''(x_1) & -\beta/m \end{pmatrix}.$$

So $\tau = -\beta/m$, $\Delta = gf''(x_1)$, $\tau^2 - 4\Delta = \beta^2/m^2 - 4gf''(x_1)$.

(1) If $f''(x_1) < 0$, it is rel. max. as a point on the graph of $f(x)$ and since $\Delta = gf''(x_1) < 0$, it is a saddle.

(2) If $f''(x_1) > 0$, then it is a rel. min. Assume $\beta > 0$. If $\beta^2/m^2 - 4gf''(x_1) > 0$, then $\tau^2 - 4\Delta = \beta^2/m^2 - 4gf''(x_1) > 0$, then we have two negative eigenvalues. Hence stable node (**overdamped**). If $\beta^2/m^2 - 4gf''(x_1) < 0$, then $\tau^2 - 4\Delta = \beta^2/m^2 - 4gf''(x_1) < 0$, complex eigenvalues with negative real part. Hence stable spiral (**underdamped**).

(3) If $f''(x_1) > 0$ and $\beta = 0$ (**undamped**), we have pure imaginary eigenvalues, so no info for nonlinear problem. However, we can use phase plane method to show it has a periodic solution. Thus the critical point is a center.

Lotka-Volterra Predator prey Model

$$\begin{aligned} x' &= -ax + bxy = x(-a + by) \\ y' &= -cxy + dy = y(-cx + d), \end{aligned}$$

Now consider the critical point $(d/c, a/b)$. A_2 has pure imaginary eigenvalues $\pm\sqrt{ad}i$. It may be a center, but need more investigation. Consider

$$\frac{dy}{dx} = \frac{y(-cx + d)}{x(-a + by)}.$$

Thus

$$\int \frac{-a + by}{y} dy = \int \frac{-cx + d}{x} dx$$

so

$$-a \ln y + by = -cx + d \ln x + c_1, \ \textit{or} \ (x^d e^{-cx})(y^a e^{-by}) = c_0.$$

We let $F(x) = x^d e^{-cx}$ and $G(x) = x^a e^{-bx}$.

Lotka-Volterra Competition Model

Two (or more) species compete for resources(food, light, etc.)(predator) of ecosystem.: **Investigate coexistence!** If x is the number of predator and y is the number of prey, then

$$\begin{aligned} x' &= \frac{r_1}{K_1} x(K_1 - x - \alpha_{12} y) \\ y' &= \frac{r_2}{K_2} y(K_2 - y - \alpha_{21} x) \end{aligned}$$

Note that the critical points are at

$$(0,0), \ (K_1, 0), \ (0, K_2) \ \textit{and} \ (\hat{x}, \hat{y}) \text{ when } \alpha_{12}\alpha_{21} \neq 0.$$

(1) If there were no second species ($y = 0$), then $x' = r_1/K_1(K_1 - x)$ so the first species grow logistically and approach the steady state(section 2)

(2) If there were no first species ($x = 0$), then $y' = r_2/K_2(K_2 - y)$ so the second species show similar behavior.

(3) The origin $(0, 0)$ is unstable.

(4) At $(\hat{x}, \hat{y})$, we see $\tau^2 - \Delta > 0$, $\tau < 0$ and $\Delta = (1 - \alpha_{12}\alpha_{21})\hat{x}\hat{y}\frac{r_1 r_2}{K_1 K_2}$. Thus

   (a) If $\alpha_{12}\alpha_{21} < 1$ then $\Delta > 0$ and we have stable node (**coexistence**)

   (b) If $\alpha_{12}\alpha_{21} > 1$ then $\Delta < 0$ and we have saddle

Example 11.4.5. Classify the critical points.

$$\begin{aligned} x' &= 0.004x(50 - x - 0.75y) \\ y' &= 0.001y(100 - y - 3.0x). \end{aligned}$$

Critical points are at $(0, 0)$, $(50, 0)$, $(0, 100)$ and $(20, 40)$. We consider $(20, 40)$. Since $\alpha_{12}\alpha_{21} = 2.25 > 1$, we have saddle.

Or you may directly compute $\mathbf{g}'((20, 40))$.

$$A_1 = \mathbf{g}'((0,0)) = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.1 \end{pmatrix}, \qquad A_3 = \mathbf{g}'((50,0)) = \begin{pmatrix} -0.2 & -0.15 \\ 0 & -0.05 \end{pmatrix}$$

$$A_3 = \mathbf{g}'((20,40)) = \begin{pmatrix} -0.08 & -0.12 \\ -0.06 & -0.04 \end{pmatrix}, \quad A_4 = \mathbf{g}'((0,100)) = \begin{pmatrix} -0.1 & 0 \\ -0.3 & -0.1 \end{pmatrix}$$

Since Δ of A_3 is negative we have saddle.

11.5 Periodic Solutions, Limit Cycles, and Global Stability

We will use the vector field, $\mathbf{V}(x, y) = (P(x, y), Q(x, y))$ to study the stability of DE.

Negative Criteria

Theorem 11.5.1 (Cycles and Critical Points)**.** *If a plane autonomous system has a periodic solution $\mathbf{x}(t)$ in a simply connected region R, then the system has at least one critical point inside the simple closed curve C. If there is a single critical point inside C, the critical point* **cannot be a saddle point**.

Corollary 11.5.2. *If a simply connected region R contains no critical point or a single saddle point, then there is no periodic solution in R.*

Example 11.5.3. Show the system

$$\begin{aligned} x' &= xy \\ y' &= -1 - x^2 - y^2 \end{aligned}$$

has no periodic solutions.

Sol. From $xy = 0$ we get $x = 0$ or $y = 0$. if $x = 0$, then from the second eq. $-1 - x^2 - y^2 = 0$, thus no critical points. By the Corollary there is no periodic solutions. The same argument shows when $y = 0$, there is no periodic solutions.

Example 11.5.4 (Lotka Volterra Competition Model)**.** The Lotka Volterra competition model

$$\begin{aligned} x' &= 0.004x(50 - x - 0.75y) \\ y' &= 0.001y(100 - y - 3.0x) \end{aligned}$$

has no periodic solutions in the first quadrant.

Sol. Critical points are at $(0, 0)$, $(50, 0)$, $(0, 100)$ and $(20, 40)$. Among them only $(20, 40)$ is in the first quadrant and it is a saddle. Hence by the above corollary, it has no periodic solutions,

Theorem 11.5.5 (Bendixon Negative Criteria)**.** *If* $\operatorname{div}\mathbf{V} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$ *does not change sign in a simply connected region* R*, then the system has no periodic solutions.*

Example 11.5.6 (Bendixon Negative Criteria)**.** Investigate periodic solutions of the following system.

$$(a)\ \begin{aligned} x' &= x + 2y + 4x^3 - y^2 \\ y' &= -x + 2y + yx^2 + y^3 \end{aligned} \qquad (b)\ \begin{aligned} x' &= y + x(2 - x^2 - y^2) \\ y' &= -x + y(2 - x^2 - y^2) \end{aligned}$$

Sol. (a) $\operatorname{div}\mathbf{V} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 1 + 12x^2 + 2 + x^2 + 3y^2 \geq 3$, so there are no periodic solutions.

(b) $\operatorname{div}\mathbf{V} = 4 - 4(x^2 + y^2)$. So if R is the interior of unit circle $x^2 + y^2 < 1$, there are no periodic solutions in the disk.

Also, if R is any simply connected region outside the disk, there are no periodic solutions since $\operatorname{div}\mathbf{V} = 4 - 4(x^2 + y^2) < 0$ in R. It follows that if there is a periodic solution, it must enclose the circle $x^2 + y^2 = 1$. In fact, one can check $x(t) = (\sqrt{2}\sin t, \sqrt{2}\cos t)$ is a periodic solution.

Example 11.5.7 (Sliding Bead)**.** Sliding Bead in Example before satisfies

$$mx'' = -mg\frac{f'(x)}{1 + [f'(x)]^2} - \beta x'. \tag{11.17}$$

Show that it has no periodic solutions.

Sol. We change it to have the following system:

$$\begin{cases} x' = y \\ y' = -g\frac{f'(x)}{1+[f'(x)]^2} - \frac{\beta}{m}y. \end{cases}$$

$\operatorname{div}\mathbf{V} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = -\frac{\beta}{m} < 0$. Hence there are no periodic solutions.

As a generalization of above theorem, we have

Theorem 11.5.8 (Dulac Negative Criteria). *If $\delta(x,y)$ is a C^1 function in a simply connected region and if* $\operatorname{div}(\delta(x,y)\mathbf{V}) = \frac{\partial(\delta P)}{\partial x} + \frac{\partial(\delta Q)}{\partial y}$ *does not change sign in a simply connected region R, then the system has no periodic solutions.*

Example 11.5.9. Show that the DE

$$x'' = x^2 + (x')^2 - x - x' \tag{11.18}$$

has no periodic solutions.

Sol. We consider the following system:

$$\begin{cases} x' &= y \\ y' &= x^2 + y^2 - x - y. \end{cases}$$

If we choose $\delta(x,y) = e^{ax+by}$ then

$$\frac{\partial(\delta P)}{\partial x} + \frac{\partial(\delta Q)}{\partial y} = e^{ax+by}(ay + 2y - 1) + e^{ax+by}b(x^2 + y^2 - x - y).$$

If we set $a = -2, b = 0$, then $\frac{\partial(\delta P)}{\partial x} + \frac{\partial(\delta Q)}{\partial y} = -e^{ax+by} < 0$. Thus there are no periodic solutions.

Positive Criteria: Poincaré-Bendixson Theory

Definition 11.5.10 (Invariant region). A region R is called an **invariant region** for an autonoumous system if whenever, $\mathbf{x}_0$ is in R, the solution $\mathbf{x}(t)$ satisfying $\mathbf{x}(0) = \mathbf{x}_0$ remains in R.

Theorem 11.5.11 (Normal vectors and invariant regions). *If $\mathbf{n}(x,y)$ is a normal vector on the boundary of R pointing inside the region, then R will be an invariant region, provided $\mathbf{V} \cdot \mathbf{n}(x,y) \geq 0$ for all points (x,y) on the boundary.*

Example 11.5.12 (Circular invariant region). Find a circular invariant region with center $(0,0)$ of the system

$$\begin{cases} x' &= -y - x^3 \\ y' &= x - y^3. \end{cases}$$

Sol. For the circle, $x^2 + y^2 = r^2$, normal vector $\mathbf{n} = (-2x, -2y)$ points towards inside the region.

Since

$$\mathbf{V} \cdot \mathbf{n} = (-y - x^3, x - y^2) \cdot (-2x, -2y) = 2(x^4 + y^4),$$

we can conclude that $\mathbf{V} \cdot \mathbf{n} \geq 0$ on the circle $x^2 + y^2 = r^2$. Therefore by the Theorem, the circular region $x^2 + y^2 \leq r^2$ is an invariant region for the system for any $r > 0$.

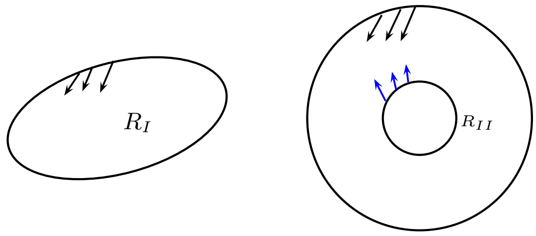

Figure 11.10: Type I and type II region for Poincaré-Bendixson Theorem

Example 11.5.13 (Annular Invariant Regions)**.** Find an annular invariant for the system

$$\begin{cases} x' &= x - y - 5x(x^2+y^2) + x^5 \\ y' &= x + y - 5x(x^2+y^2) + y^5. \end{cases}$$

Sol. The normal vector $\mathbf{n}_1 = (-2x, -2y)$ points towards inside the circle $x^2+y^2 = r^2$ while the normal vector $\mathbf{n}_2 = -\mathbf{n}_1$ is directed towards exterior. We compute

$$\mathbf{V} \cdot \mathbf{n}_1 = -2(r^2 - 5r^4 + x^6 + y^6).$$

$r^2 - 5r^4 = r^2(1-5r^2)$

If $r = 1$, $\mathbf{V} \cdot \mathbf{n}_1 = 8 - 2(x^6 + y^6) \geq 0$, since the maximum of $x^6 + y^6$ on the circle $x^2 + y^2 = 1$ is 1. The flow is directed towards the interior of the circular region.

If $r = 1/4$, $\mathbf{V}\cdot\mathbf{n}_1 \leq -2(r^2-5r^4) < 0$(Some computations) so $\mathbf{V}\cdot\mathbf{n}_2 = -\mathbf{V}\cdot\mathbf{n}_1 > 0$. The flow is directed towards the exterior of the circle $x^2 + y^2 = 1/16$. So the annular region $1/16 \leq x^2 + y^2 \leq 1$ is an invariant region for the system.

Theorem 11.5.14 (Poincaré-Bendixson - I)**.** *Let R be an invariant region for a plane autonomous system and suppose R has no critical points on its boundary.*

(a) If R is a type I region that has a single unstable node or an unstable spiral point in the interior, then there is at least one periodic solution in R.

(b) If R is a type II region that contains no critical points, then there is at least one periodic solution in R.

In either case, if $\mathbf{x}(t)$ is a nonperiodic solution in R, then $\mathbf{x}(t)$ spirals towards a cycle that is a solution to the system, call ed a limit cycle.

Example 11.5.15 (Existence of a periodic solution)**.** Show the system

$$\begin{cases} x' &= -y + x(1-x^2-y^2) - y(x^2+y^2) \\ y' &= x + y(1-x^2-y^2) + x(x^2+y^2) \end{cases}$$

has at least one periodic solution.

Sol. If $\mathbf{n}_1 = (-2x, -2y)$ is the normal vector, then $\mathbf{V} \cdot \mathbf{n}_1 = -2r^2(1-r^2)$.(Need computations) If we let $r = 2$ and $r = 1/2$ then we may conclude that $1/4 \le x^2 + y^2 \le 4$ is an invariant region for the system. If (x_1, y_1) is a critical point, then $\mathbf{V} \cdot \mathbf{n}_1 = (0,0) \cdot \mathbf{n}_1 = -2r^2(1-r^2)$. Therefor $r = 0$ or $r = 1$. If $r = 0$, $(0,0)$ is a critical point.

If $r = 1$, the system becomes $-2y = 0, 2x = 0$ thus a contradiction. Therefore, $(0,0)$ is the only critical point and it is not in R. By (b) of above theorem the system has at least on periodic solution in R.

Example 11.5.16. Van der Pol's equation. The following system has a periodic solution when $\mu > 0$.

$$y'' - \mu(1-y^2)y' + y = 0. \tag{11.19}$$

Here $\mu(1-y^2)y'$ is a damping term. If $|y| > 1$, we have positive damping and if

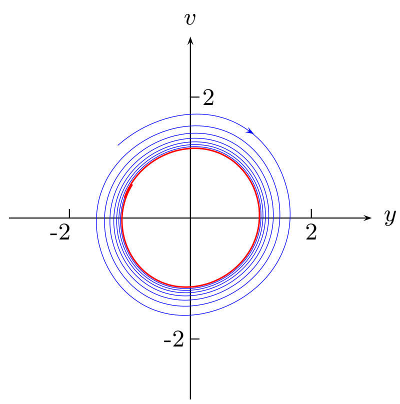

Figure 11.11: $\mu = 0.1$

$|y| < 1$, we have negative damping. With the substitution $v = y'$, $y'' = v\frac{dv}{dy}$ we get

$$v\frac{dv}{dy} - \mu(1-y^2)v + y = 0.$$

If μ is small we approximate it by $v\frac{dv}{dy} + y = 0$. So the limit cycle is close to a circle. But if μ is large, situation changes.

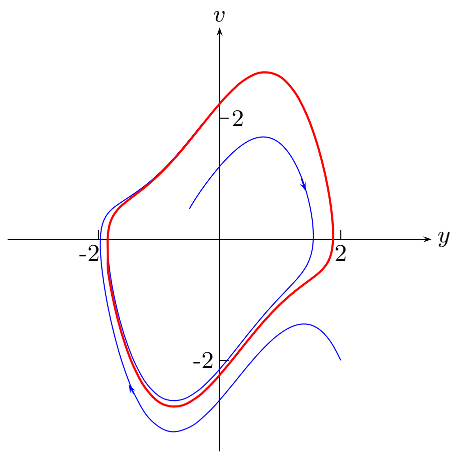

Figure 11.12: $\mu = 1.2$